Hw 5B.

1. Let $\mathfrak{M} \ni E_n \uparrow E$ (i.e. $E = \bigcup_{n=1}^{\infty} E_n$ & $E_n \subseteq E_{n+1}$ $\forall n$) show that $m(E_n) \uparrow m(E)$ (i.e. $\lim_n m(E_n) = m(E) \leq +\infty$ & $m(E_n) \leq m(E_{n+1})$ $\forall n$). Show that $\mathfrak{M} \ni E_n \downarrow E$ & $m(E) < +\infty \Rightarrow m(E_n) \downarrow m(E)$

2. Any F_σ-set K can be reprented as the union of ascending sequence of closed sets (because the union of finitely many closed sets is closed).

3. The union of countably many F_σ-sets is a F_σ-set.

4. Let $\mathcal{S}$ consist of all functions representable as $\sum_{i=1}^{N} c_i \chi_{E_i}$ with $c_i \in \mathbb{R}$ & $E_i \in \mathfrak{M}$ $\forall i$.
Let $\mathcal{S}_0 := \{ f \in \mathcal{S} : \exists E$ of measure $m(E) < +\infty$ s.t. $f = 0$ on $\mathbb{R} \setminus E \}$. Show that $\forall f \in \mathcal{S}_0$ and $\forall \varepsilon > 0$ $\exists$ a finite-length interval $[a, b]$ and step-function ψ, continuous function g from $\mathbb{R}$ to $\mathbb{R}$ vanishing outside $[a, b]$ s.t. $f = \psi$ on $\mathbb{R} \setminus A$, & $f = g$ on $\mathbb{R} \setminus A$ for some "exceptional set" A of measure $< \varepsilon$.

5. Let $f \in \mathcal{MF}(E; \mathbb{R})$ meaning that $E \in \mathcal{M}$ and $f: E \to \mathbb{R}$ and $f^{-1}(I) := \{x \in E : f(x) \in I\} \in \mathcal{M}$ for each finite-length interval I ($\in \mathcal{I}_0$ in notation). Show that $f^{-1}(I) \in \mathcal{M}$ for each interval (so $\ell(I) \leq +\infty$) and that $f^{-1}(\{y\}) \in \mathcal{M}$ $\forall y \in \mathbb{R}$.

6*. Let $E \in \mathcal{M}$ and $f: E \to \mathbb{R}$. Show that

TFSAE:

(i) $f \in \mathcal{M}(E; \mathbb{R})$.

(ii) $\{x \in E : \alpha \leq f(x)\} \in \mathcal{M}$ $\forall \alpha \in \mathbb{R}$.

(iii) $\{x \in E : \alpha < f(x)\} \in \mathcal{M}$ $\forall \alpha \in \mathbb{R}$.

(iv) $\{x \in E : \alpha \geq f(x)\} \in \mathcal{M}$ $\forall \alpha \in \mathbb{R}$.

(v) $\{x \in E : \alpha > f(x)\} \in \mathcal{M}$ $\forall \alpha \in \mathbb{R}$.

7. Let $f \in \mathcal{MF}(E;\mathbb{R})$ and $g: E \to \mathbb{R}$.

Show that

(a) If $f \sim g$ (in the sense that $f(x) = g(x)$ $\forall x \in E \setminus A$ with some $A \in \mathfrak{M}_0$ then $g \in \mathcal{MF}(E;\mathbb{R})$.

(b) $f, g \in \mathcal{MF}(E;\mathbb{R})$. Show that $f+g \in \mathcal{MF}(E;\mathbb{R})$. Hint: $\forall \alpha \in \mathbb{R}$

$$\{x \in E : \alpha < f(x) + g(x)\}$$

$$= \{x \in E : \alpha - g(x) < r < f(x) \text{ for some } r \in \mathbb{Q}$$

$$= \bigcap_{r \in \mathbb{Q}} \left(\{x \in E : \alpha - r < g(x)\} \cap \{x \in E : r < f(x)\} \right)$$

8. Let $E = E_1 \dot{\cup} E_2$ with $E_1, E_2, E \in \mathfrak{M}$.

Show that

$$f \in \mathfrak{M}\mathcal{F}(E;\mathbb{R}) \text{ iff } \left(\begin{array}{l} f|_{E_1} \in \mathfrak{M}\mathcal{F}(E_1;\mathbb{R}) \\ f|_{E_2} \in \mathfrak{M}\mathcal{F}(E_2;\mathbb{R}) \end{array} \right)$$

9. Let $E \in \mathfrak{M}$ and $f, g \in \mathfrak{M}\mathcal{F}(E;\mathbb{R})$, with

$$f : E \to (0, +\infty)$$

$$g : E \to \mathbb{R} \setminus \{0\}.$$

Show that $\frac{1}{f}, \frac{1}{g} \in \mathfrak{M}\mathcal{F}(\mathbb{R}, \mathbb{R})$.

10*. Let $E \in \mathfrak{M}$, and $f, g \in \mathfrak{M}\mathcal{F}(E;\mathbb{R})$; let $\alpha, \beta \in \mathbb{R}$. Show that

$$\alpha f + \beta g,\ f^2,\ fg \left(= \frac{(f+g)^2 - f^2 - g^2}{2} \right) \in \mathfrak{M}\mathcal{F}(E;\mathbb{R})$$

$$f \vee g \left(:= \max\{f, g\} : x \mapsto \max\{f(x), g(x)\} \right) \in \mathfrak{M}\mathcal{F}(E;\mathbb{R})$$

$$f \wedge g \in \mathfrak{M}(\mathcal{F};\mathbb{R})$$

$$|f| \in \mathfrak{M}(\mathcal{F};\mathbb{R}),$$

and

$$\frac{1}{g}, \ \frac{f}{g} \in \mathcal{MF}(E; \mathbb{R}) \text{ provided that } g(x) \neq 0 \ \forall \ x \in E.$$